

SOLITARY WAVES FOR NONLINEAR SCHRÖDINGER EQUATION WITH DERIVATIVE

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ABSTRACT. In this paper, we characterize a family of solitary waves for NLS with derivative (DNLS) by the structure analysis and the variational argument. Since (DNLS) doesn't enjoy the Galilean invariance any more, the structure analysis here is closely related with the nontrivial momentum and shows the equivalence of nontrivial solutions between the quasilinear and the semilinear equations. Firstly, for the subcritical parameters $4\omega > c^2$ and the critical parameters $4\omega = c^2, c > 0$, we show the existence and uniqueness of the solitary waves for (DNLS), up to the phase rotation and spatial translation symmetries. Secondly, for the critical parameters $4\omega = c^2, c \leq 0$ and the supercritical parameters $4\omega < c^2$, there is no nontrivial solitary wave for (DNLS). At last, we make use of the invariant sets, which is related to the variational characterization of the solitary wave, to obtain the global existence of solution for (DNLS) with initial data in the invariant set $\mathcal{K}_{\omega,c}^+ \subseteq H^1(\mathbb{R})$, with $4\omega = c^2, c > 0$ or $4\omega > c^2$.

On one hand, different with the scattering result for the L^2 -critical NLS in [10], the scattering result of (DNLS) doesn't hold for initial data in $\mathcal{K}_{\omega,c}^+$ because of the existence of infinity many small solitary/traveling waves in $\mathcal{K}_{\omega,c}^+$, with $4\omega = c^2, c > 0$ or $4\omega > c^2$. On the other hand, our global result improves the global result in [34, 35] (see Corollary 1.6).

1. INTRODUCTION

In this paper, we consider the solitary waves of nonlinear Schrödinger equation with derivative

$$\begin{cases} i\partial_t u + \partial_x^2 u + \frac{1}{2}i|u|^2 \partial_x u - \frac{1}{2}iu^2 \partial_x \bar{u} + \frac{3}{16}|u|^4 u = 0, & t \in \mathbb{R} \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \quad (1.1)$$

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the equation (1.1) appears in plasma physics [22, 23, 28], and has many equivalent forms. For example, it is equivalent to the following equation

$$\begin{cases} i\partial_t v + \partial_x^2 v + i\partial_x(|v|^2 v) = 0, & t \in \mathbb{R} \\ v(0, x) = v_0(x) \in H^1(\mathbb{R}) \end{cases} \quad (1.2)$$

by the following gauge transformation

$$v(t, x) \mapsto u(t, x) = G_{3/4}(v)(t, x) := e^{i\frac{3}{4} \int_{-\infty}^x |v(t, \eta)|^2 d\eta} v(t, x).$$

The equation (1.1) is L^2 -critical derivative NLS since the scaling transformation

$$u(t, x) \mapsto u_\lambda(t, x) = \lambda^{1/2} u(\lambda^2 t, \lambda x)$$

leaves both (1.1) and the mass invariant. The mass, momentum and energy of the solution for (1.1) are defined as following

$$M(u)(t) = \frac{1}{2} \int |u(t, x)|^2 dx, \quad (1.3)$$

$$P(u)(t) = -\frac{1}{2} \Im \int \bar{u} \partial_x u + \frac{1}{8} \int |u(t, x)|^4 dx, \quad (1.4)$$

$$E(u)(t) = \frac{1}{2} \int |\partial_x u(t, x)|^2 dx - \frac{1}{32} \int |u(t, x)|^6 dx. \quad (1.5)$$

They are conserved under the flow (1.1) by the local well-posedness theory in H^1 according to the phase rotation, spatial translation and time translation invariances. Since (1.1) or (1.2) doesn't enjoy the Galilean and pseudo-conformal invariance any more, there is no explicit blowup solution for (1.1) and the momentum is not trivial in dealing with the solitary/traveling waves of (1.1) any more.

Local well-posedness theory for (1.1) in the energy space was worked out by N. Hayashi and T. Ozawa [16, 25]. They combined the fixed point argument with $L_I^4 W_\infty^1(\mathbb{R})$ estimate to construct the local-in-time solution with arbitrary data in the energy space. For other results, we can refer to [14, 15]. Since (1.1) is \dot{H}^1 -subcritical case, the maximal lifespan interval of the energy solution only depends on the H^1 norm of initial data.

Theorem 1.1. [16, 25] *For any $u_0 \in H^1(\mathbb{R})$ and $t_0 \in \mathbb{R}$, there exists a unique maximal-lifespan solution $u : I \times \mathbb{R} \rightarrow \mathbb{C}$ to (1.1) with $u(t_0) = u_0$, the map $u_0 \rightarrow u$ is continuous from $H^1(\mathbb{R})$ to $C(I, H^1(\mathbb{R})) \cap L_{loc}^4(I; W^{1,\infty}(\mathbb{R}))$. Moreover the solution has the following properties:*

- (1) I is an open neighborhood of t_0 .
- (2) The mass, momentum and energy are conserved, that is, for all $t \in I$,

$$M(u)(t) = M(u)(t_0), \quad P(u)(t) = P(u)(t_0), \quad E(u)(t) = E(u)(t_0).$$

(3) If $\sup(I) < +\infty$, or $(\inf(I) > -\infty)$, then

$$\lim_{t \rightarrow \sup(I)} \|\partial_x u(t)\|_{L^2} = +\infty, \quad \left(\lim_{t \rightarrow \inf(I)} \|\partial_x u(t)\|_{L^2} = +\infty, \text{ respectively.} \right)$$

(4) If $\|u(0)\|_{H^1}$ is sufficiently small, then u is a global solution.

The sharp local well-posedness result in $H^s, s \geq 1/2$ is due to H. Takaoka [29] by Bourgain's Fourier restriction method. The sharpness is shown in [30] in the sense that nonlinear evolution $u(0) \mapsto u(t)$ fails to be C^3 or even uniformly C^0 in this topology, even when t is arbitrarily close to zero and H^s norm of the data is small (see also Biagioni-Linares [5]).

In [25], the global well-posedness is obtained for (1.1) in energy space under the smallness condition

$$\|u_0\|_{L^2} < \sqrt{2\pi}, \quad (1.6)$$

the argument is based on the sharp Gagliardo-Nirenberg inequality and the energy method (conservation of mass and energy). This is improved by H. Takaoka [30], who proved global well-posedness in H^s for $s > 32/33$ under the condition (1.6). His argument is based on Bourgain's restriction method, which separated the evolution of low frequencies and of high frequencies of initial data and notices that nonlinear evolution has H^1 regularity effect even for rough solution $u \in H^s$. In [8, 9], I-team used the "I-method" to show global well-posedness in $H^s, s > 1/2$ under (1.6), I-team defined Iu as a modified H^s norm, whose energy is nearly conserved in time by capturing nonlinear cancellation in frequency space under the flow (1.1). Later, Miao, Wu and Xu [21] showed the sharp global well-posedness in $H^{1/2}$ under (1.6) by using I-method and the refined resonant decomposition technique.

In this paper, we consider the existence of the solitary/traveling waves in the energy space for (1.1) and its role in the long time analysis of solution to (1.1). It is known in [27] that (1.1) has a two-parameter family of solitary/traveling waves solutions of the form:

$$u_{\omega,c}(t, x) = e^{i\omega t} \varphi_{\omega,c}(x - ct) \quad (1.7)$$

$$:= e^{i\omega t + i\frac{c}{2}(x-ct)} \phi_{\omega,c}(x - ct) \quad (1.8)$$

where $(\omega, c) \in \mathbb{R}^2$, $4\omega > c^2$ and

$$\phi_{\omega,c}(x) = \left[\frac{\sqrt{\omega}}{4\omega - c^2} \left\{ \cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}} \right\} \right]^{-1/2}, \quad (1.9)$$

which is a positive solution of

$$\left(\omega - \frac{c^2}{4}\right)\phi - \partial_x^2\phi - \frac{3}{16}|\phi|^4\phi = -\frac{c}{2}|\phi|^2\phi. \quad (1.10)$$

Note that the solitary/traveling waves have the following mass

$$\begin{aligned} \|e^{i\omega t}\varphi_{\omega,c}(x-ct)\|_{L^2}^2 &= \|e^{i\omega t + i\frac{c}{2}(x-ct)}\phi_{\omega,c}(x-ct)\|_{L^2}^2 \\ &= 8 \tan^{-1} \sqrt{\frac{2\sqrt{\omega} + c}{2\sqrt{\omega} - c}}. \end{aligned} \quad (1.11)$$

This implies

$$\lim_{4\omega > c^2, (\omega,c) \rightarrow (1,2)} \|e^{i\omega t}\varphi_{\omega,c}(x-ct)\|_{L^2} = \sqrt{4\pi}.$$

As for $(\omega, c) = (1, 0)$, the role of the momentum in (1.10) disappears. In addition, we have $E(e^{it}\varphi_{1,0}) = 0$ and $\|e^{it}\varphi_{1,0}(x)\|_{L^2} = \sqrt{2\pi}$, which corresponds to the condition (1.1) and sharp Gagliardo-Nirenberg inequality in [32]. As for $4\omega > c^2$, we have $E(e^{i\omega t}\varphi_{\omega,c}(x-ct)) < 0$ for $c > 0$, and $E(e^{i\omega t}\varphi_{\omega,c}(x-ct)) > 0$ for $c < 0$, Colin and Ohta prove its stability by the variational method (the concentration compactness argument) in [7]. For the special case $4\omega > c^2$ with $c < 0$, we can refer to [13].

As shown above, $e^{it}\varphi_{1,0}$, which corresponds to $(\omega, c) = (1, 0)$, is not the unique solitary wave of (1.1), up to the phase rotation and spatial translation symmetries. In [34], Wu showed that there exists a small $\epsilon_* > 0$, such that the solution u of (1.1) globally exists under the condition

$$\|u_0\|_{L^2} < \|\varphi_{1,0}\|_{L^2} + \epsilon_* = \sqrt{2\pi} + \epsilon_*.$$

It is the aim to characterize the solitary waves and show its role in the long time analysis of solution for (1.1) from the point of view in [26]. In order to do so, we firstly give the variational characterization of solitary waves. Now, we consider the solitary solutions for (1.1) with the following form:

$$u(t, x) = e^{i\omega t}\varphi_{\omega,c}(x-ct).$$

It is easy to verify that $\varphi_{\omega,c}$ satisfies

$$\omega\varphi - \partial_x^2\varphi - \frac{3}{16}|\varphi|^4\varphi = -ic\partial_x\varphi + \frac{1}{2}i|\varphi|^2\partial_x\varphi - \frac{1}{2}i\varphi^2\partial_x\bar{\varphi}. \quad (1.12)$$

Note that the term $-ci\partial_x\varphi + \frac{1}{2}i|\varphi|^2\partial_x\varphi - \frac{1}{2}i\varphi^2\partial_x\bar{\varphi}$ is not compatible with the momentum. While, after the key structure analysis of solution in Section 2.1, we find that

(1.12) is equivalent to the following

$$\omega\varphi - \partial_x^2\varphi - \frac{3}{16}|\varphi|^4\varphi = -ic\partial_x\varphi - \frac{c}{2}|\varphi|^2\varphi, \quad (1.13)$$

which is compatible with the mass, momentum and energy, and the solution of (1.13) is the critical point of

$$J_{\omega,c}(\varphi) := E(\varphi) + \omega M(\varphi) + cP(\varphi) \quad (1.14)$$

in $H^1(\mathbb{R})$. More precisely, we have

Theorem 1.2. *Let¹*

$$\phi_{\omega,c}(x) = \begin{cases} \left[\frac{\sqrt{\omega}}{4\omega - c^2} \left\{ \cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}} \right\} \right]^{-1/2}, & 4\omega > c^2, \\ 2\sqrt{c} \cdot (c^2x^2 + 1)^{-1/2}, & 4\omega = c^2, c > 0. \end{cases}$$

Then the following results hold

- (1) *For the subcritical case $4\omega > c^2$. $\varphi_{\omega,c}(x) = e^{i\frac{c}{2}x}\phi_{\omega,c}(x)$ is a unique solution of (1.12) in $H^1(\mathbb{R}, \mathbb{C})$, up to the phase rotation and spatial translation symmetries of (1.12).*
- (2) *For the critical case $4\omega = c^2, c > 0$. $\varphi_{\omega,c}(x) = e^{i\frac{c}{2}x}\phi_{\omega,c}(x)$ is a unique solution of (1.12) in $H^1(\mathbb{R}, \mathbb{C})$, up to the phase rotation and spatial translation symmetries of (1.12).*
- (3) *For the critical case $4\omega = c^2, c \leq 0$ and the supercritical case $4\omega < c^2$. (1.12) has no nontrivial solution in $H^1(\mathbb{R}, \mathbb{C})$.*

Remark 1.3. We make some remarks on the above result.

- (1) We have the following pointwise convergence.

$$\begin{aligned} \varphi_{\omega,c}(x) &\rightarrow 0 \quad \text{as } 4\omega > c^2, (\omega, c) \rightarrow (1, -2), \\ \varphi_{\omega,c}(x) &\rightarrow 2\sqrt{2}e^{ix}/(4x^2 + 1)^{1/2} \quad \text{as } 4\omega > c^2, (\omega, c) \rightarrow (1, 2). \end{aligned}$$

- (2) After the structure analysis (Lemma 2.1), we can also obtain the existence of the solution for (1.10) by ODE argument, (See [Theorem 5, [3]]). Here we use the variational argument to show the existence of solution for (1.12) (which is equivalent to the existence of solution for (1.10) by the structure analysis), its advantage is that we can show the global existence of the energy solution for

¹For the subcritical parameters $4\omega > c^2$, $\phi_{\omega,c}(x)$ decays exponentially, while for the critical parameters $4\omega = c^2, c > 0$, $\phi_{\omega,c}(x)$ decays polynomially.

- (1.1) in some invariant set $\mathcal{K}_{\omega,c}^+$ in Theorem 1.4 by the local wellposedness result and the variational argument.
- (3) The variational characterization of the solitary waves with $4\omega > c^2$ in [7] doesn't work for the critical case $4\omega = c^2, c > 0$ as well, we can refer to Lemma 7 in [7]. Here we use the structure analysis of solution to show the equivalence of nontrivial solution between (1.12) and (1.13). After showing this property, it is easy to use the variational method [2, 33] to show the existence of $\varphi_{\omega,c}$ to (1.13) in X_c space with structure². The uniqueness (up to the phase rotation and spatial translation symmetries)³ and H^1 regularity of the solitary wave imply the existence and uniqueness of the minimizer of the variational problem in the energy space.
- (4) About the stability of the sum of two solitary waves of (1.1) with subcritical parameters in the energy space, we can refer to [20], which is obtained by the linearized argument, modulational stability analysis and the energy method. Recently, we have learned that the stability of the sum of k solitary waves of (DNLS) has been obtained independently by Le Coz and Wu [17].

Secondly, we can consider the role of the solitary waves $e^{i\omega t}\varphi_{\omega,c}(x - ct)$ in the long time analysis of solution to (1.1). We can refer to [24, 26]. For the subcritical case $4\omega > c^2$ or the critical case $4\omega = c^2$ with $c > 0$, we let $J_{\omega,c}^0 = J_{\omega,c}(\varphi_{\omega,c})$, and introduce the functional $K_{\omega,c}(\varphi)$, which is the invariant quantity of solutions to (1.13)

$$K_{\omega,c}(\varphi) := \int \left(|\varphi_x|^2 - \frac{3}{16} |\varphi|^6 + \omega |\varphi|^2 - c \Im(\overline{\varphi} \varphi_x) + \frac{c}{2} |\varphi|^4 \right) dx,$$

and two subsets in the energy space H^1

$$\begin{aligned} \mathcal{K}_{\omega,c}^+ &:= \{ \varphi \in H^1 : J_{\omega,c}(\varphi) < J_{\omega,c}^0, \quad K_{\omega,c}(\varphi) \geq 0 \}, \\ \mathcal{K}_{\omega,c}^- &:= \{ \varphi \in H^1 : J_{\omega,c}(\varphi) < J_{\omega,c}^0, \quad K_{\omega,c}(\varphi) < 0 \}. \end{aligned}$$

As a consequence of the variational characterization of the solitary waves and the local well-posedness theory to (1.1), we have

Theorem 1.4. *The following results hold*

- (1) *For $4\omega > c^2$ or $4\omega = c^2$ with $c > 0$, we have $\mathcal{K}_{\omega,c}^\pm \neq \emptyset$, and they are invariant sets under the flow of (1.1) in $H^1(\mathbb{R}, \mathbb{C})$.*

²See definition in (2.17)

³The uniqueness is obtained by the standard ODE argument due to one dimensional spatial variable.

- (2) Let $u(0) \in H^1$, and u be the solution of (1.1) with initial data $u(0)$ and I be its maximal interval of existence. Then if $u(0) \in \mathcal{K}_{\omega,c}^+$ for some (ω, c) with $4\omega > c^2$ or $4\omega = c^2, c > 0$, then $I = \mathbb{R}$.

Remark 1.5. (1) For sufficiently small $\epsilon > 0$, we have $B_\epsilon(0) \subseteq \mathcal{K}_{\omega,c}^+$ for $4\omega > c^2$ or $4\omega = c^2$ with $c > 0$.

- (2) What will happen for the solution of (1.1) with initial data in $\mathcal{K}_{\omega,c}^+$? In fact, for $4\omega_1 > c_1^2$ with $|(\omega_1, c_1) - (1, -2)| \ll 1$ and $4\omega_2 \geq c_2^2$ with $|(\omega_2, c_2) - (1, 2)| \ll 1$, we have $M(\varphi_{\omega_1, c_1}) + \frac{c_1}{2}P(\varphi_{\omega_1, c_1}) \ll 1$ and

$$\varphi_{\omega_1, c_1} \in \mathcal{K}_{\omega_2, c_2}^+,$$

which means that there are infinity many small solitary/traveling waves in $\mathcal{K}_{\omega_2, c_2}^+$, therefore the scattering result for (1.1) with initial data in $\mathcal{K}_{\omega_2, c_2}^+$ doesn't hold any more (See Figure 1). This is a significant difference with the L^2 -critical NLS in [10].

- (3) For $4\omega > c^2$ or $4\omega = c^2$ with $c > 0$, we have no long time analysis for solutions with initial data in $\mathcal{K}_{\omega,c}^-$ since there is no effective Virial identity for the energy solution of (1.1).
- (4) In [11], Fukaya, Hayashi and Inui obtained the analogous global result for the generalized derivative nonlinear Schrödinger equation a few months after we submitted our paper.

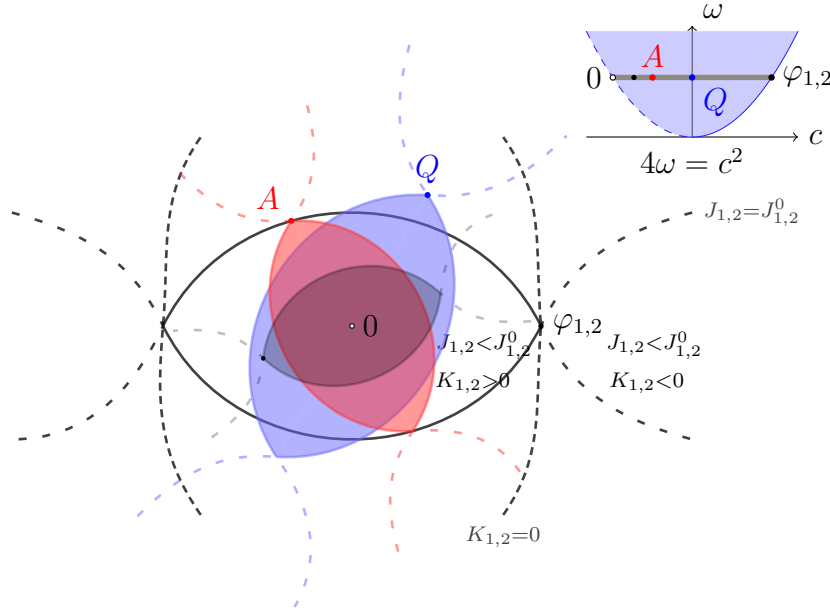


Figure 1: $\varphi_{\omega,c}$ where $Q = \varphi_{1,0}$, $A \approx \varphi_{1,-1.1}$

As a corollary, we can obtain the following result (See also Corollary 1.4 in [11]).

Corollary 1.6. *Let $u(0, \cdot) = u_0(\cdot) \in H^1(\mathbb{R})$, and satisfy one of the following conditions*

- (1) $M(u_0) < 2\pi$,
- (2) $M(u_0) = 2\pi$ and $P(u_0) < 0$,
- (3) $M(u_0) = 2\pi$ and $P(u_0) = 0$ and $E(u_0) < 0$.

Then the solution to (1.1) exists globally in $H^1(\mathbb{R})$.

Remark 1.7. This result can improve the global result in [34, 35]. In fact, we can show that the subset of $H^1(\mathbb{R})$ with the property (3) is empty by the variational characterization of the solitary wave, this phenomena is similar as that for the L^2 -critical NLS in [31].

Throughout this paper, we will use the following notations. The tempered distribution is denoted by $\mathcal{S}'(\mathbb{R}^n)$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant C . If $A \lesssim B$ and $B \lesssim A$, we say that $A \approx B$.

At last, this paper is organized as follows. In Section 2.1, we first give the structure analysis of solution to (1.12), then show the variational characterization of the solitary waves in space \tilde{H}_c for the subcritical parameters and X_c for the critical parameters with structure, and obtain the threshold $J_{\omega,c}^0$ in terms of the solitary waves in Section 2.2 and Section 2.3, respectively; In Section 3, we make use of the variational characterization of the solitary waves and the local wellposedness of (1.1) to prove Theorem 1.4 and Corollary 1.6.

2. EXISTENCE AND NONEXISTENCE OF TRAVELING WAVES

In this section, we firstly consider the existence of the solitary/traveling waves for (1.1) with the following form:

$$u(t, x) = e^{i\omega t} \varphi_{\omega,c}(x - ct).$$

It is easy to check that $\varphi_{\omega,c}$ satisfies

$$\omega\varphi - \partial_x^2\varphi - \frac{3}{16}|\varphi|^4\varphi = -ci\partial_x\varphi + \frac{1}{2}i|\varphi|^2\partial_x\varphi - \frac{1}{2}i\varphi^2\partial_x\bar{\varphi}. \quad (2.1)$$

2.1. Structure analysis, nonexistence and compactness result. Although the left hand side in (2.1) consists with the definitions of the mass and energy in (1.3) and (1.5), while the right hand side is not compatible with the definitions of the momentum in (1.4). This motivates us to explore more properties about the solitary waves. Here we make use of the structure of the solitary waves. Note that $\varphi \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\}$

is a nontrivial solution to (2.1) with the structure $\varphi(x) := e^{i\frac{c}{2}x}\phi(x)$, if and only if $\phi \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\}$ satisfies

$$\left(\omega - \frac{c^2}{4}\right)\phi - \partial_x^2\phi - \frac{3}{16}|\phi|^4\phi = -\frac{c}{2}|\phi|^2\phi + \frac{1}{2}i|\phi|^2\partial_x\phi - \frac{1}{2}i\phi^2\partial_x\bar{\phi} \quad (2.2)$$

For this equation, we have

Lemma 2.1. $\phi \in H^1(\mathbb{R}, \mathbb{C})$ is a nontrivial solution to (2.2) if and only if $\phi \in H^1(\mathbb{R}, \mathbb{C})$ satisfies

$$\left(\omega - \frac{c^2}{4}\right)\phi - \partial_x^2\phi - \frac{3}{16}|\phi|^4\phi = -\frac{c}{2}|\phi|^2\phi. \quad (2.3)$$

Proof. See Lemma 2 in [7]. □

Remark 2.2. By the proof of Theorem 8.1.6 in [6], we know that the solution of (2.3) can be taken the positive, even and real valued function up to a fixed phase rotation and spatial translation, from which we can take the solution ϕ of (2.3) to be a real function, that is $\phi \in H^1(\mathbb{R}, \mathbb{R})$.

Now we divide $(\omega, c) \in \mathbb{R}^2$ into several regions.

- (1) the supercritical case: $4\omega < c^2$;
- (2) the critical case: $4\omega = c^2$;
- (3) the subcritical case: $4\omega > c^2$.

Proposition 2.3. For the supercritical case $4\omega < c^2$ and the critical case $4\omega = c^2, c \leq 0$, (2.1) has no nontrivial solution in $H^1(\mathbb{R}, \mathbb{C})$.

Proof. After the structure analysis in Lemma 2.1 and Remark 2.2, we only need to show the nonexistence of the real valued nontrivial solution to (2.3), which can be obtained by Theorem 5 in [3]. □

Now we consider the subcritical case $4\omega > c^2$ and the critical case $4\omega = c^2, c > 0$. The special structure for ϕ implies the special structure for φ to (2.2), which induces that nontrivial solution φ to (2.2) is just the nontrivial solution φ to

$$\omega\varphi - \partial_x^2\varphi - \frac{3}{16}|\varphi|^4\varphi = -ci\partial_x\varphi - \frac{c}{2}|\varphi|^2\varphi, \quad (2.4)$$

which exactly corresponds to the definitions (1.3)-(1.5) of the mass, the momentum and the energy. Formally, φ is the critical points of the energy-mass $E + \omega M$ provided that the momentum is fixed. Since the right hand side in (2.1) or (2.4) is not semilinear,

but quasilinear, we need to combine the above structure analysis⁴, the Nehari manifold argument in [2, 33] and the symmetric-decreasing rearrangement in [18] to show the existence of the solitary waves. It also helps to give the long time analysis of solution to (1.1) in next section. By the classical argument, $\varphi \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\}$ solves (2.4) if and only if $\varphi \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\}$ is a nontrivial critical point of the following functional

$$\begin{aligned} J_{\omega,c}(\varphi) &:= E(\varphi) + \omega M(\varphi) + cP(\varphi) \\ &= \int \left(\frac{1}{2} |\partial_x \varphi|^2 - \frac{1}{32} |\varphi|^6 + \frac{\omega}{2} |\varphi|^2 - \frac{c}{2} \Im(\overline{\varphi} \partial_x \varphi) + \frac{c}{8} |\varphi|^4 \right) dx. \end{aligned} \quad (2.5)$$

It is unbounded from below in $H^1(\mathbb{R}, \mathbb{C})$. While, it is easy to check that $J_{\omega,c}$ is (at least) a C^2 functional on $H^1(\mathbb{R}, \mathbb{C})$. Moreover, as for (2.4), we consider the following quantity

$$K_{\omega,c}(\varphi) := \int \left(|\partial_x \varphi|^2 - \frac{3}{16} |\varphi|^6 + \omega |\varphi|^2 - c \Im(\overline{\varphi} \partial_x \varphi) + \frac{c}{2} |\varphi|^4 \right) dx \quad (2.6)$$

since any solution $\varphi \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\}$ to (2.4) satisfies

$$K_{\omega,c}(\varphi) = 0. \quad (2.7)$$

Before dealing with (2.4), we give a useful compactness lemma.

Lemma 2.4. *Let $1 < p < \infty$ and $\{\phi_n\}$ be a bounded sequence in $\dot{H}^1(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R})$ with radially symmetric and nonincreasing. Then, there exists $\phi \in \dot{H}^1(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R})$ with radially symmetric and nonincreasing such that for $p < q < \infty$, (up to a subsequence)*

$$\phi_n \rightarrow \phi \text{ strongly in } L^q(\mathbb{R}, \mathbb{R}).$$

Proof. Since $\phi_n \in \dot{H}^1(\mathbb{R}, \mathbb{R})$, for all n , we have by Sobolev embedding,

$$\phi_n \in C^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}).$$

Thus, we may assume that ϕ_n are well defined pointwise for all n .

Since $\{\phi_n\}$ is bounded in $\dot{H}^1(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R})$, by Sobolev embedding, if necessary up to a subsequence, there exists non-increasing, radial function $\phi \in \dot{H}^1(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R})$ such that

$$\begin{aligned} \phi_n &\rightarrow \phi \quad \text{weakly in } \dot{H}^1(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R}) \\ \phi_n &\rightarrow \phi \quad \text{strongly in } L_{\text{loc}}^q(\mathbb{R}, \mathbb{R}) \\ \phi_n &\rightarrow \phi \quad \text{a.e. on } \mathbb{R} \end{aligned} \quad (2.8)$$

⁴From the proof, the structure analysis is the key point and also necessary for us to show the existence of the solitary waves with the critical parameters $4\omega = c^2, c > 0$. The similar idea also appeared in showing the existence of the solitary/traveling waves of Gross-Pitaevskii equation in [4, 12, 19].

Using $\phi \in \dot{H}^1(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R})$, for any $\epsilon > 0$, there exists $R = R(\phi) > 1$ large enough, which is independent of n such that

$$\phi(x) < \epsilon, \text{ for any } |x| > R. \quad (2.9)$$

Moreover, by Egorov's theorem, and (2.8), there exists $E_\epsilon \subset (-2R, 2R)$ such that

$$\text{mes}((-2R, 2R) \setminus E_\epsilon) < \epsilon,$$

and

$$\phi_n(x) \rightarrow \phi(x) \quad \text{uniformly on } E_\epsilon. \quad (2.10)$$

Since $\{\phi_n\}$ is positive, radially symmetric and non-increasing, it follows from (2.9) and (2.10) that for n large enough,

$$\phi_n(x) \leq \phi_n(x_0) \leq \phi(x_0) + 2\epsilon, \quad \text{for all } x \in \mathbb{R} \setminus (-2R, 2R),$$

where $x_0 \in E_\epsilon \cap (-2R, 2R) \setminus (-R, R)$ and $\phi(x_0) < \epsilon$. Hence, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} |\phi_n(x) - \phi(x)|^q dx \\ &= \int_{(-2R, 2R)} |\phi_n(x) - \phi(x)|^q dx + \int_{\mathbb{R} \setminus (-2R, 2R)} |\phi_n(x) - \phi(x)|^q dx \\ &= \int_{E_\epsilon} |\phi_n(x) - \phi(x)|^q dx + \int_{(-2R, 2R) \setminus E_\epsilon} |\phi_n(x) - \phi(x)|^q dx \\ &\quad + \int_{\mathbb{R} \setminus (-2R, 2R)} |\phi_n(x) - \phi(x)|^q dx \\ &\leq \int_{E_\epsilon} |\phi_n(x) - \phi(x)|^q dx + \int_{(-2R, 2R) \setminus E_\epsilon} |\phi_n(x) - \phi(x)|^q dx \\ &\quad + (3\epsilon)^{q-p} \sup_n \|\phi_n\|_p^p. \end{aligned}$$

Then, for sufficiently large n , we have the result by (2.10), absolute continuity of Lebesgue integral, $\text{mes}((-2R, 2R) \setminus E_\epsilon) < \epsilon$, and the arbitrary smallness of ϵ . \square

2.2. Variational characterization for the subcritical case $4\omega > c^2$. In this subsection, we shall give the variational characterization⁵ of the solution to (2.4) in the subcritical case $4\omega > c^2$. Let $J_{\omega, c}$ and $K_{\omega, c}$ denote by (2.5) and (2.6) respectively. By

⁵In fact, Colin and Ohta[7] have given the corresponding variational characterization via the concentration compactness argument for the subcritical parameters $4\omega > c^2$, nevertheless, we will show this again by the Nehari Manifold and the non-increasing rearrangement technique. Different with Colin-Ohta's argument [7], the argument here also works for the critical parameters $4\omega = c^2, c > 0$. It will be shown in Section 2.3.

Lemma 2.1, we will consider the functional $J_{\omega,c}$ and $K_{\omega,c}$ in \tilde{H}_c . More precisely, we will consider the following Sobolev space with the rotation structure

$$\tilde{H}_c := \{ \varphi \in \mathcal{S}'(\mathbb{R}) : \varphi(x) = e^{i\frac{c}{2}x} \phi(x) \text{ with } \phi \in H^1(\mathbb{R}, \mathbb{C}) \} ,$$

with the norm

$$\|\varphi\|_{\tilde{H}_c}^2 := \|\phi\|_{H^1}^2 + \left(\omega - \frac{c^2}{4} \right) \|\phi\|_{L^2}^2, \quad \text{with } \phi \in H^1(\mathbb{R}, \mathbb{C}).$$

It is easy to verify that $(\tilde{H}_c, \|\cdot\|_{\tilde{H}_c})$ is a Hilbert space⁶. On one hand, $K_{\omega,c}$ is well defined and of class C^1 on \tilde{H}_c . On the other hand, if $\varphi \in \tilde{H}_c \setminus \{0\}$ is the solution of (2.4), then φ satisfies (2.7), which implies that $K_{\omega,c}(\varphi)$ is an invariant quantity of solutions to (2.4). Combining the above two facts, we consider the following minimization problem

$$J_{\omega,c}^0 = \inf \left\{ J_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) = 0, \varphi \in \tilde{H}_c \setminus \{0\} \right\}. \quad (2.11)$$

For convenience, we define:

$$\begin{aligned} K_{\omega,c}^Q(\varphi) &:= \int (|\partial_x \varphi|^2 + \omega |\varphi|^2 - 2c \Im(\bar{\varphi} \partial_x \varphi)) dx, \\ K_{\omega,c}^N(\varphi) &:= K_{\omega,c}^Q(\varphi) - K_{\omega,c}(\varphi) = \int \left(\frac{3}{16} |\varphi|^6 - \frac{c}{2} |\varphi|^4 \right) dx. \end{aligned}$$

By the definition, we have for $\lambda > 0$ and $\alpha \in \left(\frac{c^2}{4\omega}, 1 \right)$ ⁷

$$\begin{aligned} K_{\omega,c}^Q(\lambda\varphi) &= \lambda^2 \int (|\partial_x \varphi|^2 + \omega |\varphi|^2 - 2c \Im(\bar{\varphi} \partial_x \varphi)) dx \\ &= \lambda^2 \left[(1 - \alpha) \|\partial_x \varphi\|_{L^2}^2 + \frac{1}{\alpha} \left\| \alpha \partial_x \varphi - \frac{c}{2} i \varphi \right\|_{L^2}^2 + \left(\omega - \frac{c^2}{4\alpha} \right) \|\varphi\|_{L^2}^2 \right], \end{aligned}$$

which implies that

Lemma 2.5. *For any $\varphi \in \tilde{H}_c \setminus \{0\}$, we have*

$$\lim_{\lambda \rightarrow 0+} K_{\omega,c}^Q(\lambda\varphi) = 0.$$

The next lemma exhibits the behavior of $K_{\omega,c}$ near the origin of \tilde{H}_c .

⁶The inner product in \tilde{H}_c is induced by the inner product in $H^1(\mathbb{R}, \mathbb{C})$, and it is homeomorphic to $H^1(\mathbb{R}, \mathbb{C})$

⁷For the critical case $4\omega = c^2, c > 0$, we need take $\alpha = 1$. Notice that $K_{\omega,c}^Q(\varphi)$ is not coercive in $H^1(\mathbb{R}, \mathbb{C})$ in the critical case, it is just this difficulty which motivates us to explore the structure of the solitary waves.

Lemma 2.6. *For any bounded sequence $\{\varphi_n\} \subset \tilde{H}_c \setminus \{0\}$ with*

$$\lim_{n \rightarrow \infty} K_{\omega,c}^Q(\varphi_n) = 0.$$

We have for large n ,

$$K_{\omega,c}(\varphi_n) > 0.$$

Proof. Since $\varphi_n \in \tilde{H}_c \setminus \{0\}$, there exists $\phi_n \in H^1(\mathbb{R}, \mathbb{C})$, such that $\varphi_n(x) = e^{i\frac{c}{2}x} \phi_n(x)$ with

$$\begin{aligned} K_{\omega,c}^Q(\varphi_n) &= \int (|\partial_x \varphi_n|^2 + \omega |\varphi_n|^2 - c \Im(\overline{\varphi}_n \partial_x \varphi_n)) dx \\ &= \int \left[|\partial_x (e^{-i\frac{c}{2}x} \varphi_n)|^2 + \left(\omega - \frac{c^2}{4} \right) |e^{-i\frac{c}{2}x} \varphi_n|^2 \right] dx \\ &= \int \left[|\partial_x \phi_n|^2 + \left(\omega - \frac{c^2}{4} \right) |\phi_n|^2 \right] dx. \end{aligned}$$

By $\lim_{n \rightarrow \infty} K_{\omega,c}^Q(\varphi_n) = 0$, it follows from the Gagliardo-Nirenberg inequality that, for sufficiently large n

$$\begin{aligned} K_{\omega,c}^N(\varphi_n) &= \int \left(\frac{3}{16} |\varphi_n|^6 - \frac{c}{2} |\varphi_n|^4 \right) dx \\ &= \int \left(\frac{3}{16} |\phi_n|^6 - \frac{c}{2} |\phi_n|^4 \right) dx \\ &\lesssim \|\phi_n\|_{H^1}^2 \|\phi_n\|_{L^2}^4 + \|\phi_n\|_{H^1} \|\phi_n\|_{L^2}^3 \\ &= o(K_{\omega,c}^Q(\varphi_n)), \end{aligned}$$

where we used the fact $4\omega > c^2$. Thus, for sufficiently large n , we have

$$K_{\omega,c}(\varphi_n) = K_{\omega,c}^Q(\varphi_n) - K_{\omega,c}^N(\varphi_n) \approx K_{\omega,c}^Q(\varphi_n) > 0.$$

This completes the proof. \square

According to the aforementioned lemma, we now replace the functional $J_{\omega,c}$ (unbounded from below) in (2.11) with a positive functional $H_{\omega,c}$, while extending the minimizing region from the mountain ridge “ $K_{\omega,c} = 0$ ” to the mountain flank “ $K_{\omega,c} \leq 0$ ”. Let

$$\begin{aligned} H_{\omega,c}(\varphi) &:= J_{\omega,c}(\varphi) - \frac{1}{4} K_{\omega,c}(\varphi) \\ &= \int \left(\frac{1}{4} |\partial_x \varphi|^2 - \frac{c}{4} \Im(\overline{\varphi} \partial_x \varphi) + \frac{\omega}{4} |\varphi|^2 + \frac{1}{64} |\varphi|^6 \right) dx \\ &= \frac{1}{4} \int \left(|\partial_x (e^{-i\frac{c}{2}x} \varphi)|^2 + \left(\omega - \frac{c^2}{4} \right) |\varphi|^2 + \frac{1}{16} |\varphi|^6 \right) dx, \end{aligned} \tag{2.12}$$

which is positive. According to this definition, for any $\varphi \in \tilde{H}_c \setminus \{0\}$ and $0 < \lambda_1 < \lambda_2$, we have the following monotonicity.

$$H_{\omega,c}(\lambda_1\varphi) < H_{\omega,c}(\lambda_2\varphi). \quad (2.13)$$

In order to find the minimizers of (2.11), we turn to consider the following constrained minimization problem

$$\tilde{J}_{\omega,c}^0 = \inf \left\{ H_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) \leq 0, \varphi \in \tilde{H}_c \setminus \{0\} \right\}, \quad (2.14)$$

The following lemma shows that two minimization problems (2.11) and (2.14) are equivalent.

Lemma 2.7. *Let $J_{\omega,c}^0$ and $\tilde{J}_{\omega,c}^0$ be defined by (2.11) and (2.14) respectively. Then we have*

- (1) $J_{\omega,c}^0 = \tilde{J}_{\omega,c}^0 > 0$.
- (2) *any minimizer for (2.11) is also a minimizer for (2.14), and vice versa.*

Proof. First, by definition, we have

$$H_{\omega,c}(\varphi) = J_{\omega,c}(\varphi) \quad \text{for any } \varphi \in \tilde{H}_c \setminus \{0\} \quad \text{with } K_{\omega,c}(\varphi) = 0,$$

we have

$$\begin{aligned} J_{\omega,c}^0 &= \inf \left\{ J_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) = 0, \varphi \in \tilde{H}_c \setminus \{0\} \right\} \\ &= \inf \left\{ H_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) = 0, \varphi \in \tilde{H}_c \setminus \{0\} \right\} \\ &\geq \inf \left\{ H_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) \leq 0, \varphi \in \tilde{H}_c \setminus \{0\} \right\} \\ &= \tilde{J}_{\omega,c}^0. \end{aligned}$$

Next, for any $\varphi \in \tilde{H}_c \setminus \{0\}$ with $K_{\omega,c}(\varphi) < 0$. By Lemma 2.5 and Lemma 2.6, there exists $\lambda_0 \in (0, 1)$ such that $K_{\omega,c}(\lambda_0\varphi) = 0$. The monotonicity (2.13) of the functional $H_{\omega,c}$ implies that

$$J_{\omega,c}(\lambda_0\varphi) = H_{\omega,c}(\lambda_0\varphi) < H_{\omega,c}(\varphi).$$

Hence, we have $J_{\omega,c}^0 \leq \tilde{J}_{\omega,c}^0$, which implies (1).

Next, we show (2). On one hand, let φ be any minimizer for $\tilde{J}_{\omega,c}^0$, i.e.

$$\varphi \in \tilde{H}_c \setminus \{0\} \quad \text{with } K_{\omega,c}(\varphi) \leq 0 \text{ and } H_{\omega,c}(\varphi) = \tilde{J}_{\omega,c}^0.$$

In order to show that φ is also a minimizer for $J_{\omega,c}^0$, we only need to show that $K_{\omega,c}(\varphi) = 0$. We argue by contradiction. Assume that $K_{\omega,c}(\varphi) < 0$, by Lemma 2.5 and Lemma

2.6, there exists $\lambda_0 \in (0, 1)$ which is dependent on φ such that

$$K_{\omega,c}(\lambda_0\varphi) = 0$$

and

$$K_{\omega,c}(\lambda\varphi) < 0, \quad \text{for any } \lambda \in (\lambda_0, 1].$$

Thus by the monotonicity (2.13) of the functional $H_{\omega,c}$, we obtain that

$$\tilde{J}_{\omega,c}^0 = H_{\omega,c}(\varphi) > H_{\omega,c}(\lambda_0\varphi) = J_{\omega,c}(\lambda_0\varphi) \geq J_{\omega,c}^0 = \tilde{J}_{\omega,c}^0, \quad (2.15)$$

which is a contradiction. Hence, $K_{\omega,c}(\varphi) = 0$ and φ is also a minimizer for $J_{\omega,c}^0$. On the other hand, let φ be any minimizer for $J_{\omega,c}^0$, i.e.

$$\varphi \in \tilde{H}_c \setminus \{0\} \quad \text{with } K_{\omega,c}(\varphi) = 0, \quad \text{and } J_{\omega,c}(\varphi) = J_{\omega,c}^0.$$

Then we have

$$\tilde{J}_{\omega,c}^0 \leq H_{\omega,c}(\varphi) = J_{\omega,c}(\varphi) = J_{\omega,c}^0 = \tilde{J}_{\omega,c}^0.$$

Hence, φ is also a minimizer for $\tilde{J}_{\omega,c}^0$. This completes the proof. \square

Now, we can use the non-increasing rearrangement technique in [18] to show the existence of minimizer to (2.11).

Lemma 2.8. *There exists at least one minimizer for the minimization problem (2.11).*

Proof. Let $\{\varphi_n\} \subset \tilde{H}_c \setminus \{0\}$ be a minimizing sequence of the constrained problem (2.14), i.e.

$$K_{\omega,c}(\varphi_n) \leq 0, \quad H_{\omega,c}(\varphi_n) \geq J_{\omega,c}^0 \quad \text{and} \quad \lim_{n \rightarrow \infty} H_{\omega,c}(\varphi_n) = J_{\omega,c}^0.$$

By definition of \tilde{H}_c , there exists a sequence $\{\phi_n\} \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\}$ such that

$$\varphi_n = e^{i\frac{c}{2}x}\phi_n \quad \text{and} \quad \|\varphi_n\|_{\tilde{H}_c}^2 = \|\phi_n\|_{H^1}^2 + \left(\omega - \frac{c^2}{4}\right)\|\phi_n\|_{L^2}^2$$

Without loss of generality, we may assume that ϕ_n are real valued, radially symmetric and non-increasing about the origin of \mathbb{R} . Indeed, for any $\psi \in \tilde{H}_c \setminus \{0\}$ with $\psi = e^{i\frac{c}{2}x}\phi$ and $\|\psi\|_{\tilde{H}_c}^2 = \|\phi\|_{H^1}^2 + \left(\omega - \frac{c^2}{4}\right)\|\phi\|_{L^2}^2$, let ϕ^* be the Schwarz symmetrization of ϕ , and $\psi^* = e^{i\frac{c}{2}x}\phi^*$, by Schwarz rearrangement inequality in [18, Section 7.17], it is easy to check that

$$\begin{aligned} H_{\omega,c}(\psi) &= \frac{1}{4} \int \left(|\partial_x (e^{-i\frac{c}{2}x}\psi)|^2 + \left(\omega - \frac{c^2}{4}\right)|\psi|^2 + \frac{1}{16}|\psi|^6 \right) dx \\ &= \frac{1}{4} \int \left(|\partial_x \phi|^2 + \left(\omega - \frac{c^2}{4}\right)|\phi|^2 + \frac{1}{16}|\phi|^6 \right) dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{4} \int \left(|\partial_x \phi^*|^2 + \left(\omega - \frac{c^2}{4} \right) |\phi^*|^2 + \frac{1}{16} |\phi^*|^6 \right) dx \\
&= \frac{1}{4} \int \left(|\partial_x (e^{-i\frac{c}{2}x} \psi^*)|^2 + \left(\omega - \frac{c^2}{4} \right) |\psi^*|^2 + \frac{1}{16} |\psi^*|^6 \right) dx \\
&= H_{\omega,c}(\psi^*),
\end{aligned}$$

a similar argument shows that

$$K_{\omega,c}(\psi) \geq K_{\omega,c}(\psi^*).$$

Since $\lim_{n \rightarrow \infty} H_{\omega,c}(\varphi_n) = J_{\omega,c}^0$, we have φ_n is bounded in \tilde{H}_c , which means ϕ_n is bounded in $H^1(\mathbb{R}, \mathbb{R})$. It follows from Lemma 2.4 that there exists $\phi \in H^1(\mathbb{R}, \mathbb{R})$ such that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \phi_n &= \phi, \quad \text{weakly in } H^1(\mathbb{R}, \mathbb{R}), \\
\lim_{n \rightarrow \infty} \phi_n &= \phi, \quad \text{strongly in } L^6(\mathbb{R}, \mathbb{R}), \\
\lim_{n \rightarrow \infty} \phi_n &= \phi, \quad \text{strongly in } L^4(\mathbb{R}, \mathbb{R}).
\end{aligned}$$

Hence, from the definition of \tilde{H}_c ,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \varphi_n &= \varphi, \quad \text{weakly in } \tilde{H}_c, \\
\lim_{n \rightarrow \infty} \varphi_n &= \varphi, \quad \text{strongly in } L^6(\mathbb{R}, \mathbb{C}), \\
\lim_{n \rightarrow \infty} \varphi_n &= \varphi, \quad \text{strongly in } L^4(\mathbb{R}, \mathbb{C}),
\end{aligned}$$

where $\varphi = e^{i\frac{c}{2}x} \phi$. It follows from the weak lower continuity of the norm that

$$\begin{aligned}
H_{\omega,c}(\varphi) &\leq \lim_{n \rightarrow \infty} H_{\omega,c}(\varphi_n) = J_{\omega,c}^0 \\
K_{\omega,c}(\varphi) &\leq \liminf_{n \rightarrow \infty} K_{\omega,c}(\varphi_n) \leq 0.
\end{aligned}$$

Next, we shall prove $\varphi \neq 0$. Suppose that $\varphi = 0$, then we have

$$\begin{aligned}
0 &\leq \liminf_{n \rightarrow \infty} K_{\omega,c}^Q(\varphi_n) = \liminf_{n \rightarrow \infty} (K_{\omega,c}(\varphi_n) + K_{\omega,c}^N(\varphi_n)) \\
&\leq \liminf_{n \rightarrow \infty} K_{\omega,c}(\varphi_n) + \lim_{n \rightarrow \infty} K_{\omega,c}^N(\varphi_n) \leq 0.
\end{aligned}$$

By Lemma 2.6, there exists a subsequence φ_{n_k} such that

$$K_{\omega,c}(\varphi_{n_k}) > 0, \quad \text{for } k \text{ large enough,}$$

It is a contradiction with the choice of φ_n . Thus $\varphi \neq 0$, Hence φ is a minimizer of (2.14). By Lemma 2.7, φ is also a minimizer of (2.11). \square

Since $J_{\omega,c}$ and $K_{\omega,c}$ are C^1 functionals on \tilde{H}_c , by the above lemma, it is easy to see that if $\varphi \in \tilde{H}_c \setminus \{0\}$ is a minimizer for (2.11), then there exists $\eta \in \mathbb{R}$ such that

$$\langle J'_{\omega,c}(\varphi), \psi \rangle = \eta \langle K'_{\omega,c}(\varphi), \psi \rangle, \quad \text{for any } \psi \in \tilde{H}_c,$$

specially, if we take $\psi = \varphi$ in the above equation, then it follows from (2.7) that

$$\begin{aligned} 0 = K_{\omega,c}(\varphi) &= \int \left(|\partial_x \varphi|^2 - \frac{3}{16} |\varphi|^6 + \omega |\varphi|^2 - \frac{c}{2} \Im(\bar{\varphi} \varphi_x) + \frac{c}{2} |\varphi|^4 \right) \\ &= \eta \int \left(2 |\partial_x \varphi|^2 - \frac{9}{8} |\varphi|^6 + 2\omega |\varphi|^2 - 2c \Im(\bar{\varphi} \varphi_x) + 2c |\varphi|^4 \right) \\ &= 4\eta K_{\omega,c}(\varphi) - 2\eta \int (|\partial_x \varphi|^2 + \omega |\varphi|^2 - c \Im(\bar{\varphi} \partial_x \varphi)) - \frac{3}{8} \eta \int |\varphi|^6 \\ &= -2\eta \|\varphi\|_{\tilde{H}_c}^2 - \frac{3}{8} \eta \|\varphi\|_{L^6}^6. \end{aligned}$$

Since $\varphi \in \tilde{H}_c \setminus \{0\}$, we obtain $\eta = 0$ and

$$J'_{\omega,c}(\varphi) = 0, \quad \text{in } \tilde{H}_c^*,$$

i.e. φ satisfies (2.4) in sense of \tilde{H}_c . Since $\varphi(x) = e^{i\frac{c}{2}x} \phi(x)$, we have

$$\left(\omega - \frac{c^2}{4} \right) \phi - \partial_x^2 \phi - \frac{3}{16} |\phi|^4 \phi = -\frac{c}{2} |\phi|^2 \phi, \quad \text{in } H^1(\mathbb{R}, \mathbb{C}). \quad (2.16)$$

Note that $\phi_{\omega,c}$ in (1.9) is a solution to (2.16). By the uniqueness result (Theorem 8.1.6 in [6], ODE argument), we have

Proposition 2.9. *For subcritical case $4\omega > c^2$, up to the phase rotation and spatial translation symmetries, (2.1) has a unique solution $\varphi_{\omega,c}$ in $H^1(\mathbb{R}, \mathbb{C})$, where*

$$\varphi_{\omega,c}(x) = e^{i\frac{c}{2}x} \left[\frac{\sqrt{\omega}}{4\omega - c^2} \left\{ \cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}} \right\} \right]^{-1/2}.$$

2.3. Variational characterization for the critical case $4\omega = c^2, c > 0$. For the critical parameters $4\omega = c^2, c > 0$, the quadratic terms of the functionals $J_{\omega,c}(\varphi)$ and $K_{\omega,c}(\varphi)$ do not enjoy coercivity in H^1 , hence we can not perform the variational method (minimization) in the framework of [7] directly. Here we combine the variational method with the structure analysis to show the existence of the solitary waves to (2.4). The similar structure analysis also occurs in [4, 12, 19]. We first solve the minimization problem in the weak space X_c with structure, then show the uniqueness⁸ and the H^1 regularity of the solitary waves. Therefore we can solve the minimization problem in the energy space.

⁸Up to the phase rotation and spatial translation symmetries.

Based on the structure analysis in Section 2.1, we now consider the following space

$$X_c := \left\{ \varphi \in \mathcal{S}' : \varphi(x) = e^{i\frac{c}{2}x} \phi(x) \text{ with } \phi \in \left(\dot{H}^1 \cap L^4 \right) (\mathbb{R}, \mathbb{C}) \right\} \quad (2.17)$$

with the norm

$$\|\varphi\|_{X_c} := \|\phi\|_{\dot{H}^1} + \|\phi\|_{L^4} \quad \text{with } \phi \in \left(\dot{H}^1 \cap L^4 \right) (\mathbb{R}, \mathbb{C}).$$

It is clear that $(X_c, \|\cdot\|_{X_c})$ is a Banach space and $H^1(\mathbb{R}, \mathbb{C}) \hookrightarrow X_c$.

First, we consider the functional $J_{\omega,c}$ on X_c instead of H^1 . Similarly, it is easy to check that $J_{\omega,c}$ is (at least) a C^2 functional and unbounded from below on X_c . Moreover, $\varphi \in X_c \setminus \{0\}$ is a solution of (2.4) if and only if $\varphi \in X_c \setminus \{0\}$ is a critical point of the functional $J_{\omega,c}$.

Similarly to the subcritical case, we consider the following minimization problem

$$J_{\omega,c}^0 = \inf \{ J_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) = 0, \varphi \in X_c \setminus \{0\} \}, \quad (2.18)$$

and define

$$\begin{aligned} K_{\omega,c}^Q(\varphi) &:= \int \left(|\partial_x \varphi|^2 + \omega |\varphi|^2 - c \Im(\overline{\varphi} \partial_x \varphi) + \frac{c}{2} |\varphi|^4 \right) dx \\ &= \int \left(\left| \partial_x \left(e^{-i\frac{c}{2}x} \varphi \right) \right|^2 + \frac{c}{2} |\varphi|^4 \right) dx \geq 0, \\ K_{\omega,c}^N(\varphi) &:= K_{\omega,c}^Q(\varphi) - K_{\omega,c}(\varphi) = \frac{3}{16} \int |\varphi|^6 dx. \end{aligned}$$

By the definition, we have for $\lambda > 0$

$$K_{\omega,c}^Q(\lambda \varphi) = \lambda^2 \int \left(|\partial_x \varphi|^2 + \omega |\varphi|^2 - c \Im(\overline{\varphi} \partial_x \varphi) \right) dx + \lambda^4 \int |\varphi|^4 dx.$$

This implies that

Lemma 2.10. *For any $\varphi \in X_c \setminus \{0\}$, we have*

$$\lim_{\lambda \rightarrow 0^+} K_{\omega,c}^Q(\lambda \varphi) = 0.$$

The next lemma exhibits the behavior of $K_{\omega,c}$ near the origin of X_c .

Lemma 2.11. *For any bounded sequence $\{\varphi_n\} \subset X_c \setminus \{0\}$ with*

$$\lim_{n \rightarrow \infty} K_{\omega,c}^Q(\varphi_n) = 0.$$

We have for large n ,

$$K_{\omega,c}(\varphi_n) > 0.$$

Proof. Since $\varphi_n \in X_c \setminus \{0\}$, there exists $\phi_n \in (\dot{H}^1 \cap L^4)(\mathbb{R}, \mathbb{C})$, such that $\varphi_n(x) = e^{i\frac{c}{2}x} \phi_n(x)$ with

$$\begin{aligned} K_{\omega,c}^Q(\varphi_n) &= \int \left(|\partial_x \varphi_n|^2 + \omega |\varphi_n|^2 - c \Im(\overline{\varphi}_n \partial_x \varphi_n) + \frac{c}{4} |\varphi_n|^4 \right) dx \\ &= \int \left(|\partial_x (e^{-i\frac{c}{2}x} \varphi_n)|^2 + \frac{c}{4} |e^{-i\frac{c}{2}x} \varphi_n|^4 \right) dx \\ &= \int \left(|\partial_x \phi_n|^2 + \frac{c}{4} |\phi_n|^4 \right) dx. \end{aligned}$$

By $\lim_{n \rightarrow \infty} K_{\omega,c}^Q(\varphi_n) = 0$, it follows from the Gagliardo-Nirenberg and Hölder inequalities that

$$K_{\omega,c}^N(\varphi_n) \approx \int |\phi_n|^6 \lesssim \|\phi_n\|_{\dot{H}^1}^{2/3} \|\phi_n\|_{L^4}^{16/3} \lesssim \|\phi_n\|_{\dot{H}^1}^4 + \|\phi_n\|_{L^4}^{32/5} = o(K_{\omega,c}^Q(\varphi_n)).$$

Thus, for sufficiently large n , we have

$$K_{\omega,c}(\varphi_n) = K_{\omega,c}^Q(\varphi_n) - K_{\omega,c}^N(\varphi_n) \approx K_{\omega,c}^Q(\varphi_n) > 0.$$

This completes the proof. \square

We now replace the functional $J_{\omega,c}$ in (2.18), which is unbounded from below, with a positive functional $H_{\omega,c}$, while extending the minimizing region from “ $K_{\omega,c} = 0$ ” to “ $K_{\omega,c} \leq 0$ ”. Let

$$\begin{aligned} H_{\omega,c}(\varphi) &:= J_{\omega,c}(\varphi) - \frac{1}{6} K_{\omega,c}(\varphi) \\ &= \frac{1}{3} \int \left(|\varphi_x|^2 + \omega |\varphi|^2 - c \Im(\overline{\varphi} \partial_x \varphi) + \frac{c}{8} |\varphi|^4 \right) dx \end{aligned} \quad (2.19)$$

$$\geq 0. \quad (2.20)$$

In addition, for any $\varphi \in X_c \setminus \{0\}$ and $0 < \lambda_1 < \lambda_2$, we have the following monotonicity.

$$H_{\omega,c}(\lambda_1 \varphi) < H_{\omega,c}(\lambda_2 \varphi). \quad (2.21)$$

In order to find the minimizers of (2.18), we shall consider the following constrained minimization problem

$$\tilde{J}_{\omega,c}^0 = \inf \{ H_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) \leq 0, \varphi \in X_c \setminus \{0\} \}, \quad (2.22)$$

The following lemma shows that two minimization problems (2.18) and (2.22) are equivalent.

Lemma 2.12. *Let $J_{\omega,c}^0$ and $\tilde{J}_{\omega,c}^0$ be defined by (2.18) and (2.22) respectively. Then we have*

- (1) $J_{\omega,c}^0 = \tilde{J}_{\omega,c}^0 > 0$.
 (2) any minimizer for (2.18) is also a minimizer for (2.22), and vice versa.

Proof. First, by definition, we have

$$H_{\omega,c}(\varphi) = J_{\omega,c}(\varphi) \quad \text{for any } \varphi \in X_c \setminus \{0\} \quad \text{with } K_{\omega,c}(\varphi) = 0,$$

and

$$\begin{aligned} J_{\omega,c}^0 &= \inf \{ J_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) = 0, \varphi \in X_c \setminus \{0\} \} \\ &= \inf \{ H_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) = 0, \varphi \in X_c \setminus \{0\} \} \\ &\geq \inf \{ H_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) \leq 0, \varphi \in X_c \setminus \{0\} \} \\ &= \tilde{J}_{\omega,c}^0. \end{aligned}$$

Next, for any $\varphi \in X_c \setminus \{0\}$ with $K_{\omega,c}(\varphi) < 0$. By Lemma 2.10 and Lemma 2.11, there exists $\lambda_0 \in (0, 1)$ such that $K_{\omega,c}(\lambda_0\varphi) = 0$. The monotonicity (2.21) of the functional $H_{\omega,c}$ implies that

$$J_{\omega,c}(\lambda_0\varphi) = H_{\omega,c}(\lambda_0\varphi) < H_{\omega,c}(\varphi).$$

Hence, we have $J_{\omega,c}^0 \leq \tilde{J}_{\omega,c}^0$, which implies (1).

Next, we show (2). On one hand, let φ be any minimizer for $\tilde{J}_{\omega,c}^0$, i.e.

$$\varphi \in X_c \setminus \{0\} \quad \text{with } K_{\omega,c}(\varphi) \leq 0 \text{ and } H_{\omega,c}(\varphi) = \tilde{J}_{\omega,c}^0.$$

In order to show that φ is also a minimizer for $J_{\omega,c}^0$, we only need to show that $K_{\omega,c}(\varphi) = 0$. We argue by contradiction. Assume that $K_{\omega,c}(\varphi) < 0$, by Lemma 2.10 and Lemma 2.11, there exists $\lambda_0 \in (0, 1)$ which is dependent on φ such that

$$K_{\omega,c}(\lambda_0\varphi) = 0$$

and

$$K_{\omega,c}(\lambda\varphi) < 0, \quad \text{for any } \lambda \in (\lambda_0, 1].$$

Thus by the monotonicity (2.21) of the functional H , we obtain that

$$\tilde{J}_{\omega,c}^0 = H_{\omega,c}(\varphi) > H_{\omega,c}(\lambda_0\varphi) = J_{\omega,c}(\lambda_0\varphi) \geq J_{\omega,c}^0 = \tilde{J}_{\omega,c}^0, \quad (2.23)$$

which is a contradiction. Hence, $K_{\omega,c}(\varphi) = 0$ and φ is also a minimizer for $J_{\omega,c}^0$. On the other hand, let φ be any minimizer for $J_{\omega,c}^0$, i.e.

$$\varphi \in X_c \setminus \{0\} \quad \text{with } K_{\omega,c}(\varphi) = 0, \text{ and } J_{\omega,c}(\varphi) = J_{\omega,c}^0.$$

Then we have

$$\tilde{J}_{\omega,c}^0 \leq H_{\omega,c}(\varphi) = J_{\omega,c}(\varphi) = J_{\omega,c}^0 = \tilde{J}_{\omega,c}^0.$$

Hence, φ is also a minimizer for $\tilde{J}_{\omega,c}^0$. This completes the proof. \square

Now, we can use the non-increasing rearrangement technique in [18] once again to show the existence of minimizer to (2.18).

Lemma 2.13. *There exists at least one minimizer for the minimization problem (2.18).*

Proof. Let $\{\varphi_n\} \subset X_c \setminus \{0\}$ be a minimizing sequence of the constrained problem (2.22), i.e.

$$K_{\omega,c}(\varphi_n) \leq 0, \quad H_{\omega,c}(\varphi_n) \geq J_{\omega,c}^0 \text{ and } \lim_{n \rightarrow \infty} H(\varphi_n) = J_{\omega,c}^0.$$

By definition of X_c , there exists a sequence $\{\phi_n\} \in \left(\dot{H}^1 \cap L^4\right)(\mathbb{R}, \mathbb{C}) \setminus \{0\}$ such that

$$\varphi_n = e^{i\frac{c}{2}x} \phi_n \text{ and } \|\varphi_n\|_{X_c} = \|\phi_n\|_{\dot{H}^1} + \|\phi_n\|_{L^4}.$$

Without loss of generality, we may also assume that ϕ_n are the real valued, radially symmetric and non-increasing functions about the origin of \mathbb{R} . Indeed, for any $\psi \in X_c \setminus \{0\}$ with $\psi = e^{ix} \phi$ and $\|\psi\|_{X_c} = \|\phi\|_{\dot{H}^1} + \|\phi\|_{L^4}$, let ϕ^* be the Schwarz symmetrization of ϕ , and $\psi^* = e^{i\frac{c}{2}x} \phi^*$, by Schwarz rearrangement inequality in [18], it is easy to check that

$$\begin{aligned} H_{\omega,c}(\psi) &= \frac{1}{3} \int \left(|\partial_x \varphi|^2 + \omega |\varphi|^2 - c \Im(\bar{\varphi} \partial_x \varphi) + \frac{c}{8} |\psi|^4 \right) dx \\ &= \frac{1}{3} \int \left(|(e^{-i\frac{c}{2}x} \psi)_x|^2 + \frac{c}{8} |e^{-i\frac{c}{2}x} \psi|^4 \right) dx \\ &= \frac{1}{3} \int \left(|\partial_x \phi|^2 + \frac{c}{8} |\phi|^4 \right) dx \\ &\geq \frac{1}{3} \int \left(|\partial_x \phi^*|^2 + \frac{c}{8} |\phi^*|^4 \right) dx \\ &= H_{\omega,c}(\psi^*), \end{aligned}$$

a similar argument shows that

$$K_{\omega,c}(\psi) \geq K_{\omega,c}(\psi^*).$$

Since $\lim_{n \rightarrow \infty} H_{\omega,c}(\varphi_n) = J_{\omega,c}^0$, we have φ_n is bounded in X_c , which means ϕ_n is bounded in $\dot{H}^1(\mathbb{R}, \mathbb{R}) \cap L^4(\mathbb{R}, \mathbb{R})$. It follows from Lemma 2.4 that there exists $\phi \in \dot{H}^1(\mathbb{R}, \mathbb{R}) \cap L^4(\mathbb{R}, \mathbb{R})$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \phi_n &= \phi, \quad \text{weakly in } \dot{H}^1(\mathbb{R}, \mathbb{R}) \cap L^4(\mathbb{R}, \mathbb{R}), \\ \lim_{n \rightarrow \infty} \phi_n &= \phi, \quad \text{strongly in } L^6(\mathbb{R}, \mathbb{R}). \end{aligned}$$

Hence, from the definition of X_c ,

$$\begin{aligned}\lim_{n \rightarrow \infty} \varphi_n &= \varphi = e^{i\frac{c}{2}x} \phi, \quad \text{weakly in } X_c, \\ \lim_{n \rightarrow \infty} \varphi_n &= \varphi, \quad \text{strongly in } L^6(\mathbb{R}, \mathbb{C}).\end{aligned}$$

It follows from the weak lower continuity of the norm that

$$\begin{aligned}H_{\omega,c}(\varphi) &\leq \lim_{n \rightarrow \infty} H_{\omega,c}(\varphi_n) = J_{\omega,c}^0, \\ K_{\omega,c}(\varphi) &\leq \liminf_{n \rightarrow \infty} K_{\omega,c}(\varphi_n) \leq 0.\end{aligned}$$

Next, we shall prove $\varphi \neq 0$. Suppose that $\varphi = 0$, then we have

$$\begin{aligned}0 &\leq \liminf_{n \rightarrow \infty} K_{\omega,c}^Q(\varphi_n) = \liminf_{n \rightarrow \infty} (K_{\omega,c}(\varphi_n) + K_{\omega,c}^N(\varphi_n)) \\ &\leq \liminf_{n \rightarrow \infty} K_{\omega,c}(\varphi_n) + \lim_{n \rightarrow \infty} K_{\omega,c}^N(\varphi_n) \\ &\leq 0.\end{aligned}$$

By Lemma 2.11, there exists a subsequence φ_{n_k} such that

$$K_{\omega,c}(\varphi_{n_k}) > 0, \quad \text{for } k \text{ large enough,}$$

It is a contradiction with the choice of φ_n . Thus $\varphi \neq 0$, Hence φ is a minimizer of (2.22). By Lemma 2.12, φ is also a minimizer of (2.18). \square

Since $J_{\omega,c}$ and $K_{\omega,c}$ are C^1 functionals on X_c , by the above lemma, it is easy to see that if $\varphi \in X_c \setminus \{0\}$ is a minimizer for (2.18), then there exists $\eta \in \mathbb{R}$ such that

$$\langle J'_{\omega,c}(\varphi), \psi \rangle = \eta \langle K'_{\omega,c}(\varphi), \psi \rangle, \quad \text{for any } \psi \in X_c.$$

If we take $\psi = \varphi$ in the above equation, then it follows from (2.7) that

$$\begin{aligned}0 &= K_{\omega,c}(\varphi) \\ &= \int \left(|\partial_x \varphi|^2 - \frac{3}{16} |\varphi|^6 + \omega |\varphi|^2 - c \Im(\bar{\varphi} \partial_x \varphi) + \frac{c}{2} |\varphi|^4 \right) dx \\ &= \eta \int \left(2 |\partial_x \varphi|^2 - \frac{9}{8} |\varphi|^6 + 2\omega |\varphi|^2 - 2c \Im(\bar{\varphi} \partial_x \varphi) + 2c |\varphi|^4 \right) dx \\ &= 6\eta K_{\omega,c}(\varphi) - 4\eta \int \left(|\partial_x \varphi|^2 + \frac{c^2}{4} |\varphi|^2 - c \Im(\bar{\varphi} \partial_x \varphi) \right) dx - c\eta \int |\varphi|^4 dx \\ &= -4\eta \|\phi\|_{\dot{H}^1}^2 - c\eta \|\phi\|_{L^4}^4,\end{aligned}$$

where $\varphi(x) = e^{i\frac{c}{2}x} \phi(x)$. Since $\varphi \in X_c \setminus \{0\}$, we obtain $\eta = 0$ and

$$J'_{\omega,c}(\varphi) = 0, \quad \text{in } X_c^*,$$

i.e. φ satisfies (1.13) in sense of X_c . Since $\varphi(x) = e^{i\frac{c}{2}x}\phi(x)$, we have

$$-\partial_x^2 \phi + \frac{c}{2} |\phi|^2 \phi - \frac{3}{16} |\phi|^4 \phi = 0 \quad \text{in } \dot{H}^1(\mathbb{R}, \mathbb{C}) \cap L^4(\mathbb{R}, \mathbb{C}). \quad (2.24)$$

On the other hand, by the sharp Gagliardo-Nirenberg inequality in [1], (2.24) has a radial symmetric solution $\phi_{\omega,c}(x) = \frac{2\sqrt{c}}{\sqrt{c^2 x^2 + 1}}$. In addition, by the similar uniqueness result as Theorem 8.1.6 in [6](ODE argument), it is unique, up to the phase rotation and spatial translation symmetries. Last it is easy to verify that

$$\phi_{\omega,c} \in H^1(\mathbb{R}, \mathbb{C}). \quad (2.25)$$

It follows that

$$\|\varphi_{\omega,c}\|_{H^1}^2 = \|e^{-i\frac{c}{2}x}\phi_{\omega,c}\|_{H^1}^2 \lesssim \|\phi_{\omega,c}\|_{H^1}^2 + \|\phi_{\omega,c}\|_{L^2}^2 < \infty,$$

which means $\varphi_{\omega,c} \in H^1(\mathbb{R}, \mathbb{C}) \hookrightarrow X_c$. Thus, we have

$$\begin{aligned} J_{\omega,c}^0 &= J_{\omega,c}(\varphi_{\omega,c}) \\ &= \inf \left\{ J_{\omega,c}(\varphi) : K_{\omega,c}(\varphi) = 0, \varphi \in H^1(\mathbb{R}, \mathbb{C}) \setminus \{0\} \right\}. \end{aligned} \quad (2.26)$$

Summing up, we have

Proposition 2.14. *For the critical case $4\omega = c^2, c > 0$, up to the phase rotation and spatial translation symmetries, (2.1) has a unique solution $\varphi_{\omega,c}$ in $H^1(\mathbb{R}, \mathbb{C})$, where*

$$\varphi_{\omega,c}(x) = e^{i\frac{c}{2}x} \frac{2\sqrt{c}}{\sqrt{c^2 x^2 + 1}}.$$

Proof of Theorem 1.2. the existence and uniqueness of the solitary waves in the energy space are obtained by Proposition 2.9 and Proposition 2.14, while the nonexistence of the solitary waves in the energy space is obtain by Proposition 2.3. \square

3. GLOBAL WELL-POSEDNESS RESULT FOR SOLUTIONS WITH INITIAL DATA IN $\mathcal{K}_{\omega,c}^+$

In this section, we show Theorem 1.4 and Corollary 1.6. In order to do this, we first show the following uniformly boundedness of $K_{\omega,c}$ functional in the energy space.

Lemma 3.1. *Assume (ω, c) with $4\omega > c^2$ or $4\omega = c^2, c > 0$ and let $\varphi \in \mathcal{K}_{\omega,c}^+$, then we have*

$$K_{\omega,c}(\varphi) \geq \min \left\{ 4(J_{\omega,c}^0 - J_{\omega,c}(\varphi)), \frac{1}{4} \|e^{-i\frac{c}{2}x}\varphi\|_{H^1}^2 + \frac{1}{4} \left(\omega - \frac{c^2}{4} \right) \|\varphi\|_{L^2}^2 \right\}.$$

Proof. For the simply of notation, for any $\varphi \in \mathcal{K}_{\omega,c}^+$, we denote

$$\mathbf{j}(\lambda) := J_{\omega,c}(e^\lambda \varphi),$$

then, it is easy to see that

$$\lim_{\lambda \rightarrow -\infty} j(\lambda) = 0, \quad j'(\lambda) = K_{\omega,c}(e^\lambda \varphi),$$

and

$$\begin{aligned} & j''(\lambda) \\ &= \int \left(2e^{2\lambda} |\partial_x \varphi|^2 - 2ce^{2\lambda} \Im(\overline{\varphi} \partial_x \varphi) + 2\omega e^{2\lambda} |\varphi|^2 - \frac{9}{8} e^{6\lambda} |\varphi|^6 + 2ce^{4\lambda} |\varphi|^4 \right) dx \\ &= 4 \int \left(e^{2\lambda} |\partial_x \varphi|^2 - ce^{2\lambda} c \Im(\overline{\varphi} \partial_x \varphi) + \omega e^{2\lambda} |\varphi|^2 - \frac{3}{16} e^{6\lambda} |\varphi|^6 + \frac{c}{2} e^{4\lambda} |\varphi|^4 \right) dx \\ &\quad - \frac{3}{8} \int (e^{6\lambda} |\varphi|^6) - 2e^{2\lambda} \int (|\partial_x \varphi|^2 - c \Im(\overline{\varphi} \partial_x \varphi) + \omega |\varphi|^2) dx \\ &\leq 4j'(\lambda) - 2e^{2\lambda} \left(\|e^{-i\frac{c}{2}x} \varphi\|_{\dot{H}^1}^2 + \left(\omega - \frac{c^2}{4} \right) \|\varphi\|_{L^2}^2 \right). \end{aligned} \quad (3.1)$$

We will discuss in two cases:

Case (a): $8K_{\omega,c}(\varphi) \geq 2 \left(\|e^{-i\frac{c}{2}x} \varphi\|_{\dot{H}^1}^2 + \left(\omega - \frac{c^2}{4} \right) \|\varphi\|_{L^2}^2 \right)$. Then, we have

$$K_{\omega,c}(\varphi) \geq \frac{1}{4} \|e^{-i\frac{c}{2}x} \varphi\|_{\dot{H}^1}^2 + \frac{1}{4} \left(\omega - \frac{c^2}{4} \right) \|\varphi\|_{L^2}^2.$$

Case (b): $8K_{\omega,c}(\varphi) < 2 \left(\|e^{-i\frac{c}{2}x} \varphi\|_{\dot{H}^1}^2 + \left(\omega - \frac{c^2}{4} \right) \|\varphi\|_{L^2}^2 \right)$. By (3.1), we have for $\lambda = 0$,

$$\begin{aligned} 0 &\leq 8j'(\lambda) < 2e^{2\lambda} \left(\|e^{-i\frac{c}{2}x} \varphi\|_{\dot{H}^1}^2 + \left(\omega - \frac{c^2}{4} \right) \|\varphi\|_{L^2}^2 \right), \\ j''(\lambda) &\leq 4j'(\lambda) - 2e^{2\lambda} \left(\|e^{-i\frac{c}{2}x} \varphi\|_{\dot{H}^1}^2 + \left(\omega - \frac{c^2}{4} \right) \|\varphi\|_{L^2}^2 \right) \\ &\leq -4j'(\lambda). \end{aligned} \quad (3.2)$$

It follows from the continuity of j' and j'' with respect to λ that, j' is decreasing as λ increases until $j'(\lambda_0) = 0$ for some finite $\lambda_0 > 0$. Moreover, (3.2) holds on $[0, \lambda_0]$. By $K_{\omega,c}(e^{\lambda_0} \varphi) = 0$, we have

$$J_{\omega,c}(e^{\lambda_0} \varphi) \geq J_{\omega,c}^0.$$

Now, integrating the second inequality in (3.2), we obtain

$$-K_{\omega,c}(\varphi) = j'(\lambda_0) - j'(0) \leq -4(j(\lambda_0) - j(0)) \leq -4(J_{\omega,c}^0 - J_{\omega,c}(\varphi)).$$

This ends the proof. \square

Proof of Theorem 1.4. We first show (1). It suffices to deal with the subcritical case ($4\omega > c^2$), since the critical case ($4\omega = c^2, c > 0$) can be handled in the same way.

First of all, we define the function $j : [0, \infty) \mapsto \mathbb{R}$,

$$j(\lambda) := J_{\omega,c}(\lambda\varphi_{\omega,c}),$$

where $\varphi_{\omega,c}$ is the minimizer obtained by Lemma 2.8. On one hand, it is easy to see that

$$\lim_{\lambda \rightarrow 0+} j(\lambda) = 0.$$

Moreover, it follows from Lemma 2.7 that

$$j(1) = J_{\omega,c}^0 > 0.$$

Thus,

$$j(\lambda) < j(1) = J_{\omega,c}^0, \quad \text{for } \lambda \text{ close to zero,}$$

which means

$$J_{\omega,c}(\lambda\varphi_{\omega,c}) < J_{\omega,c}^0, \quad \text{for } \lambda \text{ close to zero.} \quad (3.3)$$

On the other hand, it follows from Lemma 2.5 and Lemma 2.6 that

$$K_{\omega,c}(\lambda\varphi_{\omega,c}) > 0 \quad \text{for } \lambda \text{ close to zero.} \quad (3.4)$$

(3.3) and (3.4) imply that

$$\lambda\varphi_{\omega,c} \in \mathcal{K}_{\omega,c}^+ \quad \text{for } \lambda \text{ close to zero,}$$

i.e. $\mathcal{K}_{\omega,c}^+ \neq \emptyset$. In the same way, by taking λ large enough, one can show that $\mathcal{K}_{\omega,c}^- \neq \emptyset$. By the variational argument of the solitary waves in Section 2 and the standard argument, we know the invariance of the sets $\mathcal{K}_{\omega,c}^+$ and $\mathcal{K}_{\omega,c}^-$ under the flow (1.1).

Next, we show (2). We define the set

$$\Omega_{\omega,c}^+ = \{t \in I : J_{\omega,c}(u(t)) < J_{\omega,c}^0, K_{\omega,c}(u(t)) \geq 0\}.$$

First, by the assumption $u_0 \in \mathcal{K}_{\omega,c}^+$, we have $0 \in \Omega_{\omega,c}^+$. Next, by the mass, energy, momentum conservation laws and the continuity of K in $H^1(\mathbb{R})$, we know that $\Omega_{\omega,c}^+(\ni 0)$ is a closed subset of I . Last by the uniform boundedness of $K_{\omega,c}$ in Lemma 3.1, we know that $\Omega_{\omega,c}^+$ is open in I . Thus we have $\Omega_{\omega,c}^+ = I$.

Now for any $t \in I$, we have

$$\int \left(|\partial_x u(t)|^2 - \frac{3}{16} |u(t)|^6 + \omega |u(t)|^2 - c \Im \left(\overline{u(t)} \partial_x u(t) \right) + \frac{c}{2} |u(t)|^4 \right) dx \geq 0,$$

which implies that

$$\int \left(|\partial_x u(t)|^2 + \omega |u(t)|^2 - c \Im \left(\overline{u(t)} \partial_x u(t) \right) \right) dx$$

$$\begin{aligned}
&\geq \int \left(\frac{3}{16} |u(t)|^6 - \frac{c}{2} |u(t)|^4 \right) dx \\
&= 4 \int \left(\frac{1}{32} |u(t)|^6 - \frac{c}{8} |u(t)|^4 \right) + \frac{1}{16} \int |u(t)|^6 dx \\
&\geq 4 \int \left(\frac{1}{32} |u(t)|^6 - \frac{c}{8} |u(t)|^4 \right) dx.
\end{aligned} \tag{3.5}$$

Note that for $t \in I$, we have

$$\begin{aligned}
J_{\omega,c}^0 &> J_{\omega,c}(u(t)) \\
&= \int \left(\frac{1}{2} |\partial_x u(t)|^2 - \frac{1}{32} |u(t)|^6 + \frac{\omega}{2} |u(t)|^2 - \frac{c}{2} \Im \left(\overline{u(t)} \partial_x u(t) \right) + \frac{c}{8} |u(t)|^4 \right) dx \\
&= \frac{1}{2} \int \left(|\partial_x u(t)|^2 + \omega |u(t)|^2 - c \Im \left(\overline{u(t)} \partial_x u(t) \right) \right) - \int \left(\frac{1}{32} |u(t)|^6 - \frac{c}{8} |u(t)|^4 \right) dx.
\end{aligned}$$

By (3.5) and the Cauchy inequality, we obtain

$$\begin{aligned}
J_{\omega,c}^0 &\geq \frac{1}{4} \int \left(|\partial_x u(t)|^2 + \omega |u(t)|^2 - c \Im \left(\overline{u(t)} \partial_x u(t) \right) \right) dx \\
&\geq \frac{1}{4} \left(\frac{1}{2} \int |\partial_x u(t)|^2 dx + (\omega - 2c^2) \int |u(t)|^2 dx \right) \\
&= \frac{1}{4} \left(\frac{1}{2} \int |\partial_x u(t)|^2 dx + (\omega - 2c^2) \int |u_0|^2 dx \right),
\end{aligned}$$

which implies $\|u(t)\|_{\dot{H}^1}$ is uniformly bounded on I , thus $I = \mathbb{R}$ by the local wellposedness theory (Theorem 1.1), which completes the proof. \square

Proof of Corollary 1.6. By the assumptions, there exists some $c \gg 1$ such that for $M(u_0) \neq 0$,

$$\begin{aligned}
J_{c^2/4,c}(u_0) &\triangleq E(u_0) + \frac{c^2}{4} M(u_0) + cP(u_0) \\
&< \frac{c^2}{4} 2\pi = J_{c^2/4,c}^0(\varphi_{c^2/4,c}), \\
K_{c^2/4,c}(u_0) &\triangleq \int |\partial_x u_0|^2 - \frac{3}{16} |u_0|^6 + \frac{c^2}{4} |u_0|^2 - c \Im(\overline{u_0} \partial_x \varphi) + \frac{c}{2} |u_0|^4 dx \\
&> 0,
\end{aligned}$$

it implies that $u(0) \in \mathcal{K}_{c^2/4,c}^+$ for some $c \gg 1$. Therefore we obtain the result by Theorem 1.4. \square

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REFERENCES

- [1] Agueh, M.: Sharp Gagliardo-Nirenberg Inequalities and Mass Transport Theory, *J. Dyn. Differ. Equat.* 18, 1069–1093(2006).
- [2] Ambrosetti, A., Malchiodi, A.: *Nonlinear analysis and semilinear elliptic problems*, Cambridge Studies in Advanced Mathematics, 104, Cambridge University Press (2007).
- [3] Berestycki, H., Lions P. L.: Nonlinear scalar field equations. I: Existence of a ground state. *Arch. Ration. Mech. Anal.* 82, 313–345(1983).
- [4] Béthuel, F., Gravejat P., Saut J. C.: Traveling waves for the Gross-Pitaevskii equation II, *Comm. Math. Phys.* 285, 567–651(2009).
- [5] Biagioni, H., Linares, F.: Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations, *Trans. Amer. Math. Soc.* 353 (9), 3649–3659(2001).
- [6] Cazenave, T.: *Semilinear Schrödinger equations*, Courant Lecture Notes Math., Vol 10, New York University, Courant Institute of Mathematical Sciences, New York, Amer. Math. Soc., Providence, RI (2003).
- [7] Colin, M., Ohta, M.: Stability of solitary waves for derivative nonlinear Schrödinger equation, *Ann. I. H. Poincaré-AN*, 23, 753–764(2006).
- [8] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Global well-posedness result for Schrödinger equations with derivative, *SIAM J. Math. Anal.* 33(2), 649–669(2001).
- [9] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: A refined global well-posedness result for Schrödinger equations with derivatives, *SIAM J. Math. Anal.* 34, 64–86(2002).
- [10] Dodson B., Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state, [arXiv:1104.1114](https://arxiv.org/abs/1104.1114).
- [11] Fukaya, N., Hayashi, M., Inui, T., Global well-posedness on a generalized derivative nonlinear Schrödinger equation, [arXiv:1610.00267](https://arxiv.org/abs/1610.00267).
- [12] Gérard, P.: The Gross-Pitaevskii equation in the energy space, Stationary and time dependent Gross-Pitaevskii equations, *Contemp. Math.*, 473, Amer. Math. Soc., Providence, RI, 129–148(2008).
- [13] Guo, B., Wu, Y.: Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation, *J. Differ. Equat.* 123, 35(1995).
- [14] Hayashi, N.: The initial value problem for the derivative nonlinear Schrödinger equation in the energy space, *Nonl. Anal.* 20, 823–833(1993).
- [15] Hayashi, N., Ozawa, T.: On the derivative nonlinear Schrödinger equation, *Physica D.* 55, 14–36(1992).
- [16] Hayashi, N., Ozawa, T.: Finite energy solution of nonlinear Schrödinger equations of derivative type, *SIAM J. Math. Anal.* 25, 1488–1503(1994).
- [17] Le Coz, S., Wu, Y.: Stability of multi-solitons for the derivative nonlinear Schrödinger equation, Preprint.

- [18] Lieb E. H., Loss, M.: Analysis, Second Edition, GSM, Vol. 14, Amer. Math. Soc., Providence, Rhode Island (2001).
- [19] Maris, M.: Traveling waves for nonlinear Schrödinger equations with nonzero condition at infinity, *Ann. Math.* 178, 107–182(2013).
- [20] Miao C., Tang X., Xu G., Stability of the traveling waves for the derivative Schrödinger equation in the energy space, to appear in *Calculus of Variations and Partial Differential Equations*.
- [21] Miao, C., Wu Y., Xu, G.: Global well-posedness for Schrödinger equation with derivative in $H^{\frac{1}{2}}(\mathbb{R})$, *J. Differ. Equat.* 251, 2164–2195(2011).
- [22] Mio, W., Ogino, T., Minami, K., Takeda, S.: Modified nonlinear Schrödinger for Alfvén waves propagating along the magnetic field in cold plasma, *J. Phys. Soc. Japan*, 41, 265–271(1976).
- [23] Mjølhus, E.: On the modulational instability of hydromagnetic waves parallel to the magnetic field, *J. Plasma Phys.* 16, 321–334(1976).
- [24] Nakanishi K., Schlag W., Invariant manifolds and dispersive Hamiltonian evolution equations, *Zurich Lectures in Advanced Mathematics*. European Mathematical Society (EMS), Zürich, 2011.
- [25] Ozawa, T.: On the nonlinear Schrödinger equations of derivative type, *Indiana Univ. Math. J.* 45, 137–163(1996).
- [26] Payne L. E., Sattinger D. H., Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.*, 22:3-4, 273–561(1975).
- [27] Saarloos, W. V., Hohenberg, P. C.: Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations, *Physica D.* 56, 303-367(1992).
- [28] Sulem C., Sulem, P.-L.: The nonlinear Schrödinger equation: Self-focusing and wave collapse *Appl. Math. Sci.*, 139, Springer-Verlag (1999).
- [29] Takaoka, H.: Well-posedness for the one dimensional Schrödinger equation with the derivative nonlinearity, *Adv. Differ. Equat.* 4 , 561–680(1999).
- [30] Takaoka, H.: Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces, *Electron. J. Differ. Equat.* 42, 1–23(2001).
- [31] Weinstein, M. I.: On the structure and formation of singularities in solutions to nonlinear dispersive evolution equations, *Comm. PDEs*, 11(5), 545–565(1986).
- [32] Weinstein, M. I.: Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.* 87(4), 567–576(1983).
- [33] Willem, M.: Minimax theorems, Birkhauser, Boston (1996).
- [34] Wu, Y.: Global well-posedness of the derivative nonlinear Schrödinger equations in energy space, *Analysis & PDE.* 6 (8), 1989–2002(2013).
- [35] Y. Wu, Global well-posedness on the derivative nonlinear Schrödinger equation revisited, [arXiv:1404.5159](https://arxiv.org/abs/1404.5159).

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